

Some half-BPS solutions of M-theory

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ABSTRACT: It was recently shown that half BPS-solutions of M-theory can be expressed in terms of a single function satisfying the 3d continuum Toda equation. In this note half-BPS solutions corresponding to separable solutions of the Toda equation are examined.

KEYWORDS: String theory, M-theory, BPS.

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1. Introduction

It was recently shown by Lin, Lunin and Maldacena (LLM) that half BPS-solutions of M-theory can be expressed in terms of a single function D satisfying the 3d continuum Toda equation[1]. Solutions to this equation are not easily found. There is however a simple class of solutions which turns out to lead to physically interesting geometries, namely separable solutions which can locally be expressed in terms of a quadratic form in a single variable and solutions of the Liouville equation.

Solutions of 11-dimensional supergravity are of great interest and have been discussed extensively in the literature (see for example [2]-[7] and references therein). The approach described in [1] provides a way to study them in a unified framework. The moduli space of half-BPS solutions is of great interest from the point of view of the AdS/CFT correspondence[8]. Already some very interesting physical questions (such as topology change [9]) have been addressed using this approach (in particular the relation to the phase space of free fermions[10]).

The LLM description applies to both regular and singular solutions. For a regular solution the function D has to satisfy specific boundary conditions. Generic solutions for D will not satisfy these boundary conditions and will give rise to singular geometries, which may be interpreted as a sign that important degrees of freedom have been neglected.

It turns out that among the geometries corresponding to separable solutions of the Toda equation there is only one non-singular example – the Maldacena-Nunez solution[6], which describes the near-horizon limit of M5-branes wrapping holomorphic curves in Calabi-Yau threefolds. All the other possibilities which stem from

separable solutions of the Toda equation turn out to have singularities if standard signature is imposed.

Two special cases are discussed in this note. One is the Alishahiha-Oz solution[3], which describes a system of intersecting M5-branes. This solution is a warped product of AdS_5 and a 6-dimensional space and as such is dual to a superconformal field theory in four dimensions. The other example discussed below is a warped product with an AdS_2 factor, which is expected to be dual to a model of superconformal quantum mechanics. This geometry describes the near-horizon limit of a system of intersecting M2-branes[5]. Warped products with AdS factors are of great interest from various points of view, including the AdS/CFT correspondence as well as flux compactifications to four dimensions.

This note begins by citing the relevant formulas from [1], after which the separable solutions of the Toda equations are described. Rather than analyzing all the cases systematically, only some general features are given and a few special cases are discussed, possibly leaving a more complete presentation for the future.

2. LLM solutions

The general solution of the supergravity equations of motion described by LLM has the form

$$ds_{11}^2 = -4e^{2\lambda}(1 + y^2e^{-6\lambda})(dt + V_i dx^i)^2 + 4e^{2\lambda}d\Omega_5^2 + y^2e^{-4\lambda}d\tilde{\Omega}_2^2 + \frac{e^{-4\lambda}}{1 + y^2e^{-6\lambda}}[dy^2 + e^D(dx_1^2 + dx_2^2)] \quad (2.1)$$

$$e^{-6\lambda} = \frac{\partial_y D}{y(1 - y\partial_y D)} \quad (2.2)$$

$$V_i = \frac{1}{2}\epsilon_{ij}\partial_j D, \quad (2.3)$$

where $i, j = 1, 2$. This, supplemented by the fluxes given in [1], provides an M-theory background preserving 16 of the original 32 supersymmetries.

The function D which determines the solution obeys the equation

$$(\partial_{x_1}^2 + \partial_{x_2}^2)D + \partial_y^2 e^D = 0. \quad (2.4)$$

This is the 3-dimensional continuous version of the Toda equation. To obtain regular supergravity solutions one has to impose specific boundary conditions on D so that potential singularities inherent in (2.1) do not occur. These conditions can be found in [1].

Note that the form of the ansatz is preserved under y -independent conformal transformations of the $x_1 - x_2$ plane provided D is shifted appropriately:

$$x_1 + ix_2 \rightarrow g(x_1 + ix_2), \quad D \rightarrow D - \log |\partial g|^2. \quad (2.5)$$

The solutions examined above can be Wick-rotated (as discussed by LLM) to yield solutions of 11-dimensional supergravity which contain an AdS_5 factor and a compact 6-dimensional manifold. These solutions can be interpreted as dual to conformal field theories in four dimensions. Specifically, one can get solutions of the form of an AdS_5 warped product by performing the analytic continuation

$$\psi \rightarrow \tau \quad \alpha \rightarrow i\rho . \quad (2.6)$$

This maps

$$\cos^2 \alpha d\psi^2 + d\alpha^2 + \sin^2 \alpha d\Omega_3^2 \rightarrow -(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2) \quad (2.7)$$

i.e.

$$d\Omega_5^2 \rightarrow -ds_{AdS_5}^2 . \quad (2.8)$$

In addition one has to take[1]

$$\lambda = \tilde{\lambda} + i\frac{\pi}{2} , \quad (2.9)$$

with the remaining coordinates unchanged. For real $\tilde{\lambda}$ one finds a metric with the correct signature¹. This way one arrives at

$$ds_{11}^2 = e^{2\tilde{\lambda}} \left(4ds_{AdS_5}^2 + y^2 e^{-6\tilde{\lambda}} d\tilde{\Omega}_2^2 + ds_4^2 \right) , \quad (2.10)$$

where

$$ds_4^2 = 4(1 - y^2 e^{-6\tilde{\lambda}})(d\chi + V_i dx^i)^2 + \frac{e^{-6\tilde{\lambda}}}{1 - y^2 e^{-6\tilde{\lambda}}} [dy^2 + e^D (dx_1^2 + dx_2^2)] \quad (2.11)$$

and

$$e^{-6\tilde{\lambda}} = -\frac{\partial_y D}{y(1 - y\partial_y D)} , \quad (2.12)$$

where D satisfies the Toda equation (2.4). Note that due to the analytic continuation regular solutions in this case have to satisfy different boundary conditions than those for (2.1). The form (2.10) characterizes all M-theory compactifications to AdS_5 which preserve $N = 2$ supersymmetry in four dimensions.

Instead of the Wick rotation (2.6), (2.9) one can instead do

$$y \rightarrow iy , \quad x_k \rightarrow ix_k , \quad (2.13)$$

which leads to metrics of the form of a warped product of AdS_2 with a 9-dimensional manifold:

$$ds_{11}^2 = e^{2\tilde{\lambda}} \left(4ds_{S^5}^2 + y^2 e^{-6\tilde{\lambda}} ds_{AdS_2}^2 - ds_4^2 \right) , \quad (2.14)$$

where ds_4^2 and $e^{-6\tilde{\lambda}}$ are as above. The form (2.14) characterizes all M-theory compactifications to AdS_2 which preserve $N = 2$ supersymmetry in four dimensions.

¹Since t is now a spacelike coordinate, it is denoted by χ below to avoid confusion.

3. Separable solutions of the Toda equation

Separable solutions of (2.4) are of the form²:

$$D(x_1, x_2, y) = F(x_1, x_2) + G(y) . \quad (3.1)$$

Using this in the Toda equation (2.4) one finds that

$$G(y) = \log(\alpha y^2 + \beta y + \gamma) , \quad (3.2)$$

while F has to satisfy the Liouville equation:

$$(\partial_{x_1}^2 + \partial_{x_2}^2)F + 2\alpha e^F = 0 . \quad (3.3)$$

Here α , β and γ are constants.

In terms of $\xi \equiv x_1 + ix_2$ one has the well known general solution of the Liouville equation[12]:

$$e^F = \frac{4|f'(\xi)|^2}{(1 + \alpha|f(\xi)|^2)^2} , \quad (3.4)$$

where f is a holomorphic function.

Thus the function D is given by

$$D(x_1, x_2, y) = \log \frac{4Q(y)|f'(\xi)|^2}{(1 + \alpha|f(\xi)|^2)^2} \quad (3.5)$$

with

$$Q(y) = \alpha y^2 + \beta y + \gamma . \quad (3.6)$$

The parameters α, β, γ parameterize the possible solutions. It is convenient to discuss separately the following cases:

1. $\alpha = 0$, i.e. $Q = \beta y + \gamma$;
2. $\alpha < 0$ and $Q = -|\alpha|(y - y_1)(y - y_2)$ for real y_1, y_2 ;
3. $\alpha > 0$ and $Q = \alpha(y - y_1)(y - y_2)$ for real y_1, y_2 ;
4. $\alpha > 0$ and $Q = \alpha|y - u|^2$ for complex u .

and it will be assumed that $y_1 < y_2$. In the first case β can be scaled away by redefining the function f in (3.5). In the remaining cases α can be scaled away in the same way once its sign is fixed.

Since the Toda equation is invariant under

$$\xi \longrightarrow g(\xi) \quad (3.7)$$

²In a different context such solutions were discussed in [13, 11].

for any holomorphic function g , one can locally choose a convenient canonical form for the function G . In view of this, one can take the following solutions for D :

$$D_0(x_1, x_2, y) = \log(y + \gamma) , \quad (3.8)$$

$$D_1(x_1, x_2, y) = \log \frac{(y - y_1)(y_2 - y)}{x_2^2} , \quad (3.9)$$

$$D_2(x_1, x_2, y) = \log \frac{4(y - y_1)(y - y_2)}{(1 + (x_1^2 + x_2^2)^2)^2} , \quad (3.10)$$

$$D_3(x_1, x_2, y) = \log \frac{4|y - u|^2}{(1 + (x_1^2 + x_2^2)^2)^2} . \quad (3.11)$$

4. M-theory solutions

Each of the solutions (3.8)–(3.11) for D leads to an $N = 2$ supersymmetric solution of the supergravity equations of motion. However to interpret the resulting metrics in physical terms one has to properly define the ranges of coordinates so that the signature is correct. All but one of the metrics arising from (3.8)–(3.11) are singular at some points. Such singularities are assumed to indicate that some degrees of freedom which generically decouple become light[3, 4] and have to be accounted for if a non-singular description is to ensue.

A case which is regular was already pointed out in [1], where the following solution of the Toda equation is discussed:

$$e^D = \frac{1}{x_2^2}(1 - 4y^2) . \quad (4.1)$$

This leads to the regular geometry found earlier by Maldacena and Nunez[6]. The form (4.1) is clearly separable and is in fact of the form D_1 given in (3.9). Substituting the solution (3.9) in (2.10) yields (note that $y_1 < y < y_2$):

$$\begin{aligned} ds^2 = & (y_1 y_2 - y^2)^{1/3} \left(4 \left(\frac{y}{y_1 + y_2 - 2y} \right)^{1/3} ds_{AdS_5}^2 + \frac{y^{4/3}(y_1 + y_2 - 2y)^{2/3}}{y_1 y_2 - y^2} d\Omega_2^2 + \right. \\ & + 4 \left(\frac{y}{y_1 + y_2 - 2y} \right)^{1/3} \frac{(y - y_1)(y - y_2)}{y_1 y_2 - y^2} \left(d\chi - \frac{dx_2}{x_2} \right)^2 + \\ & \left. + \left(\frac{y_1 + y_2 - 2y}{y} \right)^{2/3} \left(\frac{dy^2}{(y - y_1)(y - y_2)} - \frac{dx_1^2 + dx_2^2}{x_2^2} \right) \right) . \end{aligned} \quad (4.2)$$

This metric is singular if the roots y_1, y_2 are arbitrary. However if one sets $y_1 = -s, y_2 = s$ (for some real s), then the metric is regular (once χ is identified with period 2π):

$$\begin{aligned} ds^2 = & 2^{2/3}(s^2 + y^2)^{1/3} \left(2 ds_{AdS_5}^2 + \frac{y^2}{s^2 + y^2} d\Omega_2^2 \right. \\ & \left. + 2 \frac{s^2 - y^2}{s^2 + y^2} \left(d\chi - \frac{dx_1}{x_2} \right)^2 + \frac{dy^2}{s^2 - y^2} + \frac{dx_1^2 + dx_2^2}{x_2^2} \right) . \end{aligned} \quad (4.3)$$

By rescaling y one can reduce the s -dependence to an overall factor. Substituting $y = s \cos \theta$ one finds the metric

$$ds^2 = (2s)^{2/3} \Delta^{1/3} \left(2ds_{AdS_5}^2 + \Delta^{-1} \cos^2 \theta d\Omega_2^2 + \right. \\ \left. + 2\Delta^{-1} \sin^2 \theta \left(d\chi - \frac{dx_1}{x_2} \right)^2 + d\theta^2 + \frac{dx_1^2 + dx_2^2}{x_2^2} \right) , \quad (4.4)$$

where $\Delta = 1 + \cos^2 \theta$, which is up to an overall factor the same as the metric appearing in [6]. The metric (4.4) has the form of an AdS_5 fibration, and so is expected to be the supergravity dual of a superconformal field theory in four dimensions defined on the boundary of AdS_5 . It may be interesting to study the singular deformations of (4.4) described by (4.3).

Another previously known case is a singular solution first discussed in [3] as the description of the near-horizon limit of a system of intersecting M5-branes. In the present context it arises by using solution D_0 , eq. (3.9), in the metric (2.10). To obtain the correct signature one needs $\gamma > 0$ and $-\gamma < y < 0$. The metric can be written as

$$ds^2 = -\gamma^{1/3} y^{1/3} \left(4ds_{AdS_5}^2 - \frac{y}{\gamma} d\Omega_2^2 + \frac{y+\gamma}{\gamma} d\chi^2 - \frac{dy^2}{y(y+\gamma)} - \frac{dx_1^2 + dx_2^2}{y} \right) . \quad (4.5)$$

The change of variable $y = -\gamma \sin^2 \alpha$ results in

$$ds^2 = 4\gamma^{2/3} \sin^{2/3} \alpha \left(ds_{AdS_5}^2 + \frac{1}{4} \sin^2 \alpha d\Omega_2^2 + d\alpha^2 + \cos^2 \alpha d\chi^2 + \frac{dx_1^2 + dx_2^2}{\gamma \sin^2 \alpha} \right) , \quad (4.6)$$

which is of the form given in [3]. The singularity at $\alpha = 0$ was interpreted there as being due to M2-branes ending on the M5-branes.

A somewhat similar example arises from using solution D_0 (3.8) in the metric (2.14). To have the correct signature one needs $\gamma < 0$ and $y > |\gamma|$. The metric reads

$$ds^2 = \frac{y^{4/3}}{\gamma^{2/3}} \left(ds_{AdS_2}^2 + 4|\gamma| d\Omega_5^2 + \frac{|\gamma|}{y^2} (dx_1^2 + dx_2^2) + \right. \\ \left. + 4 \frac{y - |\gamma|}{y} d\chi^2 + \frac{|\gamma|}{y^2(y - |\gamma|)} dy^2 \right) . \quad (4.7)$$

Setting $\gamma = -1$ and $y = 1/\sin^2 \alpha$ leads to

$$ds^2 = \sin^{-8/3} \alpha \left(ds_{AdS_2}^2 + d\alpha^2 + \cos^2 \alpha d\chi^2 + \sin^2 \alpha d\Omega_5^2 + \sin^4 \alpha (dx_1^2 + dx_2^2) \right) , \quad (4.8)$$

which is the form found in [5] for the near-horizon geometry of a system of semilocalised intersecting M2-branes. As this is an example of an AdS_2 fibration it can be expected to be dual to a superconformal quantum mechanics.

5. Conclusions

Although very simple, separable solutions of the Toda equation lead to a large family of half-BPS solutions of M-theory which include at least some physically interesting cases. All but one of the solutions arising this way are singular. It would be interesting to establish whether these geometries have interpretations in terms of branes. It could also be of interest to explore these solutions more closely, in particular, to understand the origin and interpretation of their singularities. Another natural question is whether there is a geometric interpretation of parameters appearing in the quadratic form Q .

It is perhaps disappointing that this class of supergravity solutions does not include any regular cases beyond the well-known Maldacena-Nunez solution. To find new regular cases one has to understand more general (in particular non-separable) solutions of the Toda equation. One example of a non-separable solution is in fact determined by the functions D_3 in eq. (3.11) if one allows the parameter u appearing there to depend holomorphically on $\xi \equiv x_1 + ix_2$. Such solutions were introduced and studied by Calderbank and Tod[13]. While non-separable, these solutions also do not lead to regular metrics. It appears that for this purpose the construction described by Ward[14] (which is discussed by LLM), may be more promising.

References

- [1] H. Lin, O. Lunin and J. Maldacena, “Bubbling AdS space and 1/2 BPS geometries,” *JHEP* **0410**, 025 (2004) [arXiv:hep-th/0409174].
- [2] J. P. Gauntlett, “Intersecting branes,” arXiv:hep-th/9705011.
- [3] M. Alishahiha and Y. Oz, “AdS/CFT and BPS strings in four dimensions,” *Phys. Lett. B* **465**, 136 (1999) [arXiv:hep-th/9907206].
- [4] Y. Oz, “Warped compactifications and AdS/CFT,” arXiv:hep-th/0004009.
- [5] M. Cvetič, H. Lu, C. N. Pope and J. F. Vazquez-Poritz, “AdS in warped spacetimes,” *Phys. Rev. D* **62**, 122003 (2000) [arXiv:hep-th/0005246].
- [6] J. M. Maldacena and C. Nunez, “Supergravity description of field theories on curved manifolds and a no Int. J. Mod. Phys. A **16**, 822 (2001) [arXiv:hep-th/0007018].
- [7] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, “Supersymmetric AdS(5) solutions of M-theory,” *Class. Quant. Grav.* **21**, 4335 (2004) [arXiv:hep-th/0402153].
- [8] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1999)] [arXiv:hep-th/9711200].
- [9] P. Horava and P. G. Shepard, “Topology changing transitions in bubbling geometries,” *JHEP* **0502**, 063 (2005) [arXiv:hep-th/0502127].
- [10] D. Berenstein, “A toy model for the AdS/CFT correspondence,” *JHEP* **0407**, 018 (2004) [arXiv:hep-th/0403110].
- [11] S. V. Ketov, “Gravitational dressing of D-instantons,” *Phys. Lett. B* **504**, 262 (2001) [arXiv:hep-th/0010255].
- [12] Jürgen Jost, “Compact Riemann surfaces : an introduction to contemporary mathematics” 2nd ed. Springer 2002.
- [13] D. M. J. Calderbank and P. Tod, “Einstein metrics, hypercomplex structures and the Toda field equation,” *Differ. Geom. Appl.* **14**, 199 (2001) [arXiv:math.dg/9911121].
- [14] R. S. Ward, “Einstein-Weyl Spaces And SU(Infinity) Toda Fields,” *Class. Quant. Grav.* **7**, L95 (1990).